2.9 Dimension and Rank

Recall a **basis** for a subspace H of $\mathbb{R}^n$ is a linearly independent set in H that spans H.

Coordinate Systems

Suppose the set $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace H. For each $\mathbf{x}$ in H, the **coordinates of $\mathbf{x}$ relative to the basis $\mathcal{B}$** are the weights $c_1, \ldots, c_p$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$, and the vector in $\mathbb{R}^p$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is called the **coordinate vector of $\mathbf{x}$ (relative to $\mathcal{B}$)** or the **$\mathcal{B}$-coordinate vector of $\mathbf{x}$**.

Example 1. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then $\mathcal{B}$ is a basis for $H = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$ because $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

ANS: $\vec{x}$ is in $H = \text{Span}\{\vec{v}_1, \vec{v}_2\} \iff \vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$ for some c_1, c_2.

$\iff$ The system $c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is consistent.

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 3 \\ 0 = 0 \end{cases}$$

Thus $\vec{x} = 2\vec{v}_1 + 3\vec{v}_2$, i.e. $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

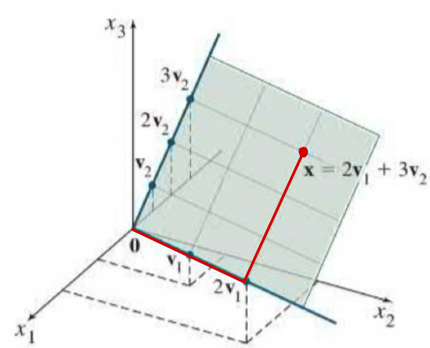

FIGURE 1 A coordinate system on a plane H in $\mathbb{R}^3$.

The Dimension of a Subspace

The **dimension** of a nonzero subspace H, denoted by $\dim H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\mathbf{0}\}$ is defined to be zero.

Example 2. The echelon form of A is given, find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$ and then state the dimensions of these subspaces.

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ANS: By Thm 13 in §2.8, a basis for Col A is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \right\}. \qquad \dim(\operatorname{Col} A) = 3.$$

For Nul A, we solve $A\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 9 & 5 & 4 & 0 \\ 0 & 1 & -3 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 3x_3 - 7x_5 = 0 \\ x_4 - 2x_5 = 0 \\ 0 = 0 \end{cases} \qquad \Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Thus a basis for Nul A is

$$\left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}. \qquad \text{Thus } \dim(\operatorname{Nul} A) = 2.$$

The **rank** of a matrix A, denoted by rank A, is the dimension of the column space of A.

$\dim(\text{Col } A) = \text{rank } A$

Theorem 14 The Rank Theorem

If a matrix A has n columns, then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.

$\dim(\text{Col } A)$

$\dim(H) = p$

Theorem 15 The Basis Theorem

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.

Example 3. Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

H

$$\begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -7 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ 9 \\ -5 \end{bmatrix}$$

$H = \text{Col} A$

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ -1 & -3 & 2 & 4 & -8 \\ -2 & -1 & -6 & -7 & 9 \\ 5 & 6 & 8 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 3 & -6 & -9 & 15 \\ 0 & -4 & 8 & 12 & -20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & -1 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for H is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \\ 6 \end{bmatrix} \right\},$$

so $\dim H = 2$

Rank and the Invertible Matrix Theorem

Theorem The Invertible Matrix Theorem (continued)
Let A be an $n \times n$ matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.
m. The columns of A form a basis of $\mathbb{R}^n$.
n. $\text{Col}\, A = \mathbb{R}^n$
o. $\text{rank}\, A = n$
p. $\dim \text{Nul}\, A = 0$
q. $\text{Nul}\, A = \{\mathbf{0}\}$

Example 4. If the rank of a 9×8 matrix A is 7, what is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

By Thm 14 (the Rank Thm) $\underbrace{\text{rank}A}_{\shortparallel\ 7} + \dim(\text{Nul}\,A) = 8 \Rightarrow \dim(\text{Nul}\,A) = 8-7=1$.

Thus the dimension of the solution space of $A\vec{x}=\vec{0}$ is 1.

Exercise 5. Suppose a 4×6 matrix A has four pivot columns. Is $\text{Col}\, A = \mathbb{R}^4$? Is Nul $A = \mathbb{R}^2$? Explain your answers.

- Col $A = \mathbb{R}^4$. since A has a pivot in each row. and so the columns of A span $\mathbb{R}^4$.
- Nul A cannot equal to $\mathbb{R}^2$. because Nul A is a subspace of $\mathbb{R}^6$

Exercise 6. Construct a 5×3 matrix with rank 2.

A rank 2 matrix has a 2-dimensional column space.

One such matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$